Int J Theor Phys (2008) 47: 1979–1986
DOI 10.1007/s10773-007-9641-6

Massive Radiation via Tunneling in a BTZ Black Hole

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Received: 24 October 2007 / Accepted: 28 December 2007 / Published online: 15 January 2008

Abstract Hawking radiation can usefully be viewed as a semi-classical tunneling process that originates at the black hole horizon. Massive radiation from a BTZ black hole is investigated. The conservation of energy implies the effect of self-gravitation. Viewed as a tunneling process, the emission spectrum derivates from the pure thermal spectrum, but it is consistent with an underlying unitary theory. The result is the same as that of massless particles.

Keywords $2+1$-dimensional BTZ black hole · Quantum tunneling · Hawking radiation · Bekenstein-Hawking entropy · Unitary theory · Painleve coordinates

1 Introduction

The fascinating properties of the black hole, classical and especially quantum, have long made it desirable to have a lower-dimensional analog which could exhibit the key features without the unnecessary complication [1–7]. Stephen Hawking's astounding discovery [8] that black holes radiate thermally set up a disturbing and difficult problem: what happens to information during black hole evaporation? Hawking's result implies the loss of unitary [9]. Although results in string theory support the idea that Hawking radiation can be described within a manifestly unitary theory [10], it remains a mystery how information is returned. In recent several years, Parikh and Wilczek proposed a method to calculate the emission rate at which particles tunnel across the event horizon [11–13]. They found that the barrier is created by the outgoing particle itself, actually it is due to the effect of self-gravitation. The method leads to nonthermality of the emission, and may plausibly lead to short-time correlations in the spectrum. Following this method, Hemming and Keski-Vakkuri have investigated the radiation from AdS black holes [14], and Medved have studied those from a de Sitter cosmological horizon [15]. After that, Zhang has extended the research from static

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spherical spacetime to stationary Kerr and Kerr-Newman black holes [16, 17]. In all these works, they only investigated the tunneling of the massless particles which go along the null geodesics. Thinking of a massive particle's emission, [18] treated the massive particle as a de Broglie s-wave. Moreover, massive charged particle emission was studied in [19]. Till now, all the results are consistent with an underlying unitary theory, and the tunneling rate can be written as the exponent of the difference in the Bekenstein-Hawking entropy, ΔS_{BH}, before and after emission.

Considering of spinning and charged BTZ black holes, since they are similar as Kerr or Reissner-Nordstrom black holes, we expect to get the same results. The tunneling effect of massless particles from BTZ black holes have been investigated in [20–22]. However, massive particles have not been considered. In this paper, massive particles tunneling rate from BTZ black holes will be calculated, and the result will be compared with that of massless particles.

The remainder of the paper is organized as follows. In Sect. 2, the line element of the spinning BTZ black hole will be given, and a Painleve-BTZ coordinate system will be introduced. In Sect. 3, on the base of Sect. 2, we will calculate the tunneling rate of massive particles. In Sect. 4, massive tunneling from a charged BTZ black hole will be investigated. In the last section, some conclusions and discussions will be given.

2 Spinning BTZ Black Hole and Painleve Coordinate System

The BTZ black holes [1, 2, 5, 23, 24] are solutions of the standard Einstein-Maxwell equation in $2+1$ spacetime dimensions, with a negative cosmological constant. Ignoring the Maxwell field, we can write down the action of a three dimensional theory of gravity as

$$I = \frac{1}{2\pi}\int \sqrt{-g}[R + 2l^{-2}]d^2x dt + B, \tag{1}$$

where B is a surface term and the radius l is related to the cosmological constant by $\Lambda = -l^{-2}$. A black hole solution has been found for the action (1) as following

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2\left(d\varphi - \frac{J}{2r^2}dt\right)^2, \tag{2}$$

with M the ADM mass, J the angular momentum (spin) of the BTZ black hole and $-\infty < t < +\infty$, $0 \le r < +\infty$, $0 \le \varphi \le 2\pi$.

For the positive mass black hole spectrum with spin ($J \neq 0$), the line element (2) has two horizons:

$$r_\pm^2 = l^2\left\{\frac{M}{2}\left[1 \pm \left(1 - \left(\frac{J}{Ml}\right)^2\right)^{1/2}\right]\right\}. \tag{3}$$

Of these, r_+, r_- are the black hole outer and inner horizon respectively. In order for the horizon to exist one must have $M > 0$, $|J| \le Ml$. In the extremal case $|J| = Ml$, both roots of $g_{00} = 0$ coincide.

The area A_H and Hawking temperature T_H of the event (outer) horizon are [3, 25]

$$A_H = 2\pi l\left\{\frac{M}{2}\left[1 + \left(1 - \left(\frac{J}{Ml}\right)^2\right)^{1/2}\right]\right\}^{1/2} = 2\pi r_+, \tag{4}$$

$$T_H = \frac{\sqrt{2}}{2\pi l}\frac{\sqrt{M^2 - J^2/l^2}}{(M + \sqrt{M^2 - J^2/l^2})^{1/2}} = \frac{1}{2\pi l^2}\left(\frac{r_+^2 - r_-^2}{r_+}\right). \tag{5}$$

The entropy of the spinning BTZ black hole is

$$S_{BH} = 4\pi r_+, \tag{6}$$

and if we reinstate the Planck units (since in BTZ units $8\hbar G = 1$) we can get

$$S_{BH} = \frac{1}{4\hbar G} A_H,$$

which is the well-known Bekenstein-Hawking area formula (S_{BH}) for the entropy, and has been proven by counting excited states in [26].

With black hole emission in mind, we now begin a semi-classical calculation that is based on the Kraus-Wilczek treatment [27–29]. To do a tunneling computation at the event horizon, we should find a coordinate system that is well-behaved there. At first, we investigate the dragging coordinate system. Let

$$\frac{d\varphi}{dt} = \frac{J}{2r^2} = \Omega, \tag{7}$$

then the line element of BTZ black hole can be rewritten as

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2. \tag{8}$$

In fact, the metric (8) represents a 2-dimensional hypersurface in 3-dimensional BTZ spacetime. This dragging coordinate system is not the final one that we want to use for resolving tunneling effect in BTZ spacetime. We need another transformation to make none of the components of either the metric or the contra metric diverge at the horizon. Moreover, constant time slices are just flat Euclidean in polar coordinate. To obtain a coordinate system analogous to Painleve coordinates [30], we should perform a coordinate transformation $dt = d\tau + f(r)dr$, where $f(r)$ is a function of r, independent on t.

Let $f(r)$ satisfy

$$f^2(r) = \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1}\left[\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} - 1\right], \tag{9}$$

then, (8) can be changed into

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)d\tau^2 + 2\left[-\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) + 1\right]^{1/2} d\tau dr + dr^2. \tag{10}$$

There is now no singularity at the event horizon, and the true character of the spacetime, as being stationary but not static due to r-direction tunneling, is manifest.

The null geodesics described by (10) are as following

$$\dot{r} = \frac{dr}{d\tau} = \pm 1 - \left[1 - \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)\right]^{1/2}, \tag{11}$$

where a dot means derivation with respect to τ, and the $+(-)$ sign can be identified with outgoing (ingoing) polar motion, respectively, under the assumption that τ increases towards future.

3 Tunneling Process of Massive Particles

It is well known that a massive particle does not follow polar null geodesics (11). According to de Broglie hypothesis, the outgoing particle shell should be considered as a sort of de Broglie s-wave. The wave function under WKB approximation is

$$\psi(r,\tau) = Ce^{i(\int_{r_i-\varepsilon}^{r} p_r dr - \omega\tau)}, \tag{12}$$

where $r_i - \varepsilon$ represents the initial location of the particle. If we let $\int_{r_i-\varepsilon}^{r} p_r dr - \omega\tau = \phi_0$, we have the phase velocity of de Broglie wave

$$\frac{dr}{d\tau} = \dot{r} = \frac{\omega}{k}, \tag{13}$$

where k is the de Broglie wave number which comes from p_r. Unlike the electromagnetic wave, the phase velocity v_p of de Broglie wave is not equal to the group velocity v_g. The definitions and relations between them are [18]

$$v_p = \frac{dr}{d\tau} = \dot{r} = \frac{\omega}{k}, \qquad v_g = \frac{dr_c}{d\tau} = \frac{d\omega}{dk}, \qquad v_p = \frac{1}{2}v_g, \tag{14}$$

where r_c denotes the location of the tunneling particle.

To obtain the formula of the phase velocity $\dot{r}$, let's now investigate the behavior of a massive particle tunneling across the event horizon. Since tunneling across the barrier is an instantaneous process, there are two simultaneous events during the process. One is particle tunneling into the barrier, another is particle tunneling out the barrier. In terms of Landau's theory of the coordinate clock synchronization, the difference of coordinate times of these two simultaneous events is $d\tau = -\frac{g_{0i}}{g_{00}}dx^i = -\frac{g_{01}}{g_{00}}dr_c, d\varphi = 0$. The group velocity is

$$v_g = \frac{dr_c}{d\tau} = -\frac{g_{00}}{g_{01}}, \tag{15}$$

and the phase velocity is therefore

$$\dot{r} = v_p = \frac{1}{2}v_g = -\frac{1}{2}\frac{g_{00}}{g_{01}} = \frac{1}{2}\frac{(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})}{[-(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}) + 1]^{1/2}}. \tag{16}$$

Moreover, if the self-gravitation are included, (10), (16) all should be changed by $M \to M - \omega$, where ω is the particle's energy.

We evaluate the imaginary part of the action for an outgoing positive energy particle which crosses the horizon outwards from:

$$r_{in}^2 = r_+^2(M,l,J) = l^2\left\{\frac{M}{2}\left[1 + \left(1 - \left(\frac{J}{Ml}\right)^2\right)^{1/2}\right]\right\}, \tag{17}$$

to

$$r_{out}^2 = r_+^2(M-\omega,l,J) = l^2\left\{\frac{M-\omega}{2}\left[1 + \left(1 - \left(\frac{J}{(M-\omega)l}\right)^2\right)^{1/2}\right]\right\}. \tag{18}$$

The imaginary part of the action is

$$\operatorname{Im} I = \operatorname{Im}\int_{r_{in}}^{r_{out}} p_r dr = \operatorname{Im}\int_{r_{in}}^{r_{out}}\int_0^{p_r} dp'_r dr. \tag{19}$$

We make the transition from the momentum variable to the energy variable using Hamilton's equation $\dot{r} = \frac{dH}{dp_r}$. Thinking about (16) in the vicinity of event horizon, the result is

$$\operatorname{Im} I = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{2[-(-(M-\omega') + \frac{r^2}{l^2} + \frac{J^2}{4r^2}) + 1]^{1/2}(-d\omega')dr}{-(M-\omega') + \frac{r^2}{l^2} + \frac{J^2}{4r^2}}, \tag{20}$$

where the minus sign of $d\omega'$ is due to the Hamiltonian being equal to the modified mass $H = M - \omega$. This is not disturbing since $r_{in} > r_{out}$. After some calculations (involving contour integration into the lower half of ω' plan), we can get $\operatorname{Im} I = 2\pi(r_{in} - r_{out})$. Apparently the emission rate depends not only on the mass M and angular momentum (spin) J of BTZ black hole but also on the energy ω of the emitted massive particle

$$\Gamma(\omega, M, l, J) \sim e^{-2\operatorname{Im} I} = \exp[4\pi(r_{out} - r_{in})]. \tag{21}$$

Comparing (6) and (21), we have

$$\Gamma(\omega, M, l, J) = \exp(\Delta S_{BH}). \tag{22}$$

4 Massive Tunneling from a Charged BTZ Black Hole

The model of interesting, $2+1$-dimensional AdS-Maxwell gravity, can be described by the following gravitational action

$$I_G = \frac{1}{4}\int d^3x\sqrt{-g}\left[\frac{1}{4\pi G}(R - 2\Lambda) - F^{\mu\nu}F_{\mu\nu}\right], \tag{23}$$

where G is the 3-dimensional Newton constant, $\Lambda = -l^{-2}$ is the negative cosmological constant. Note that we are assuming vanishing rotation for the sake of simplicity [5].

A static, charged black hole solution for the above action can be expressed as follows [1, 2, 4, 5, 23]

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2 d\varphi^2, \tag{24}$$

where

$$N^2(r) = \frac{r^2}{l^2} - 8GM - Q^2 \ln\left(\frac{r^2}{l^2}\right), \tag{25}$$

here, M is the generalized ADM mass of the charged BTZ black hole, and Q is the dimensionless parameter that represents the charge.

Usually it is more convenient to re-express (25) as

$$N^2(r) = \frac{r^2}{l^2} - \frac{r_+^2}{l^2} - Q^2 \ln\left(\frac{r^2}{r_+^2}\right), \tag{26}$$

where r_+ is the black hole horizon, which is also the outmost value of r satisfying $N^2(r_+) = 0$. Typically, there will exist some second value $r_- \le r_+$ such that $N^2(r_-) = 0$ as well.

To obtain the condition of extremality, $r_- = r_+$, let us consider the Hawking temperature (T_H) as determined by the surface gravity (κ) at the horizon [31]

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{dN^2}{dr}\bigg|_{r=r_+} = \frac{r_+}{2\pi l^2}\left(1 - \frac{l^2 Q^2}{r_+^2}\right). \tag{27}$$

It follows that the black hole becomes extremal when T_H vanishes:

$$(r_+)^2_{ext} = l^2 Q^2, \tag{28}$$

while the two roots of $N^2(r) = 0$ coincide. Then the value of $N^2(r)$ at the horizon can be written as

$$N^2(r) = Q^2 - 8GM - Q^2 \ln Q^2 = -8GM + Q^2[1 - \ln Q^2]. \tag{29}$$

Now, the function $Q^2[1 - \ln Q^2]$ vanishes at $Q = 0$, has a maximum at $Q^2 = 1$ with value 1, vanishes at $Q^2 = e$ and tends to negative infinity for larger Q^2. This means that if $8GM > 1$ there are always two roots $r_\pm$ which are different. When $8GM = 1$ the two roots coincide for $Q^2 = 1$. If $0 < 8GM < 1$ there are two branches where $N^2(r) = 0$ can have two roots or to coincide into one.

The Bekenstein-Hawking entropy of the charged BTZ black hole can be written as

$$S = 4\pi r_+ = \frac{1}{4G\hbar} A_H, \tag{30}$$

noting $8G\hbar = 1$ in BTZ $2 + 1$-dimensional spacetime.

Using Painleve-like coordinates [30], the transformation can be written as $d\tau = dt - \frac{\sqrt{1-N^2(r)}}{N^2(r)} dr$. Substituting this expression into (24), we have

$$ds^2 = -N^2(r)d\tau^2 + dr^2 + 2\sqrt{1 - N^2(r)}d\tau dr + r^2 d\varphi^2, \tag{31}$$

where the constant-time slices are just flat Euclidean in polar coordinate. It is also easy to see that the components of the metric (31) satisfy Landau's theory of the coordinate clock synchronization $\frac{\partial}{\partial x^j}(-\frac{g_{0i}}{g_{00}}) = \frac{\partial}{\partial x^i}(-\frac{g_{0j}}{g_{00}})$. This makes the quantum tunneling as an instantaneous process to be possible [19].

According to de Broglie hypothesis, the phase velocity is therefore

$$\dot{r} = v_p = \frac{1}{2} v_g = -\frac{1}{2}\frac{g_{00}}{g_{01}} = \frac{1}{2}\frac{N^2(r)}{\sqrt{1 - N^2(r)}}. \tag{32}$$

Thinking of (32) in the vicinity of event horizon, the imaginary part of the action is as the following

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_{in}}^{r_{out}} p_r dr = \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp'_r dr = \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{2\sqrt{1 - N^2(r)}(-d\omega')dr}{N^2(r)}. \tag{33}$$

Finishing above integral yields

$$\mathrm{Im}\, I = -\frac{1}{2}[4\pi r_+(M - \omega, Q) - 4\pi r_+(M, Q)] = -\frac{1}{2}\Delta S_{BH}, \tag{34}$$

where $\Delta S_{BH} = S_{BH}(M-\omega, Q) - S_{BH}(M, Q)$ is the difference of the entropies of the black hole before and after the emission. So, we have

$$\Gamma \sim \exp[-2\mathrm{Im}\, I] = e^{\Delta S_{BH}}. \tag{35}$$

5 Conclusions and Discussions

The emission spectrum (22), (35) are consistent with an underlying unitary theory, and they take the same functional form as that of massless particle [32]. Moreover, these equations also imply that the spectrum is not precisely thermal. To compare with the pure thermal spectrum, after expanding ΔS_{BH} in ω, and neglecting the quadratic terms and higher order terms of ω, then we have $\Delta S_{BH} = -\beta\omega$, where β is the inverse temperature. That is, the corrected spectrum is not precisely thermal. Only the leading-order term gives the thermal Boltzmann factor. The nonthermality suggests the possibility of information carrying correlation in the radiation.

Quantum mechanics tells us that the rate must be expressed as

$$\Gamma(i \to f) = |M_{fi}|^2 \cdot (\text{phase space factor}),$$

where the first term on the right is the square of the amplitude for the process. The phase space factor is obtained by summing over final states and averaging over initial states. But the number of final states is just the exponent of the final entropy, while the number of initial states is the exponent of the initial entropy. Hence

$$\Gamma = \frac{e^{S_{final}}}{e^{S_{intial}}} = \exp(\Delta S_{BH}), \tag{36}$$

which is in agreement with our results in (22) and (35). This suggests that the formula we have is actually exact, up to a prefactor.

Acknowledgements This work is supported by the National Natural Science Foundation of China (Grant No.10773002) and the National Basic Research Program of China (Grant No. 2003CB716302).

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